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Published in:
Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, CDC/CCC 2009

Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Stability of quantized time-delay nonlinear systems: A Lyapunov-Krasowskii-functional approach

Claudio DE PERSIS, Frédéric MAZENC

Abstract— **Lyapunov-Krasowskii functionals are used to design quantized control laws for nonlinear continuous-time systems in the presence of time-invariant constant delays in the input. The quantized control law is implemented via hysteresis to avoid chattering. Under appropriate conditions, our analysis applies to stabilizable nonlinear systems for any value of the quantization density. The resulting quantized feedback is parametrized with respect to the quantization density. Moreover, the maximal allowable delay tolerated by the system is characterized as a function of the quantization density.**

I. Introduction

Quantized control systems ([4], [9]), are systems in which the control law is a piece-wise constant function of time taking values in a finite set. The design of quantized control systems is based on a partition of the state space. One value of the control law is associated to each set of the partition, and whenever the state crosses the boundary between two sets of the partition, the control law takes the new value associated to the set which the state has just entered.
When dealing with the problem of stabilizing the origin of the state space for linear discrete-time systems, the paper [4] has shown the effectiveness of logarithmic quantization in which the partition of the state space is coarser away from the origin and denser in its vicinity. It has also introduced the notion of quantization density, that is the number of regions of the partition per unit of space. Intuitively, the larger is the quantization density, the easier is the quantized control problem, since as the quantization density gets larger, the quantized control law approaches a control law without quantization. The paper [9] deals with a similar problem but for nonlinear continuous-time systems which can be made input-to-state stable with respect to the quantization error. Recently, the paper [1] has investigated quantized control systems in the framework of discontinuous control systems, discussing appropriate notions of solutions, namely Krasowskii and Carathéodory solutions. In this framework, the effect of quantization is viewed as an additional disturbance whose effect is attenuated by a Lyapunov redesign of the control law. Namely, given any nonlinear continuous-time process which is stabilizable by a continuous feedback, and given any value of the quantization density, it is always possible to find a new feedback depending on the quantization density, in such a way that the process in closed-loop with the quantized control law is practically stable with a basin of attraction which can be made arbitrarily large. Other notions of robustness (namely, robustness in the sense of the $\mathcal{L}_2$-gain) in connection with quantized control problems have been examined in [6] and [1]. Moreover, in the former, an adaptive quantized control scheme has been investigated.
Since quantized controls take values in a finite set, they lend themselves to be implemented over a finite data-rate communication channel. Data transmitted over a channel are usually delivered at the other end of channel after a delay. The problem of quantized control systems in the presence of delays then arises very naturally. Such a problem has been examined for the first time in [10], where the connection between Razumikhin-type theorems and the ISS small-gain theorem established in [17] was exploited. In recent years, besides [17], other contributions in the area of nonlinear time-delay systems have appeared (see, for instance, [13], [14], [12], [15], [7], [8], [11], [5] and references therein). In particular, the paper [11] has proposed a Lyapunov-Krasowskii-functional approach to study the stabilizability of nonlinear systems in the presence of a delay in the input.

The aim of this paper is to pursue the approach of [11] in the analysis and design of quantized time-delay control systems. Besides the use of Lyapunov-Krasowskii functionals, there are other important features of the approach which make our paper different from other contributions. We implement the quantized control with the hysteretic mechanism suggested in [6] to avoid chattering. It is known from [2] that, in the case no delay is present, the analysis of such hysteretic solutions can be reduced to the analysis of Krasowskii and Carathéodory solutions considered in [1]. In the case of quantized time-delay systems, the adoption of the hysteretic solution is desirable. First, because it allows us to avoid technical issues related to more general notions of solutions of time-delay quantized (that is, discontinuous) systems. Second, the existence of more general solutions such as Carathéodory solutions is guaranteed only under additional conditions (see e.g. [1]). Another feature which is worth mentioning is that, as in [1], our analysis applies to stabilizable nonlinear systems for any value of the quantization density, provided that suitable conditions are satisfied. Then, the quantized feedback which stabilizes the closed-loop system despite the delay turns out to be parametrized with respect to the quantization density.
Our approach leads to a set of conditions to design quantized control systems which are robust with respect to delays.

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978-1-4244-3872-3/09/$25.00 ©2009 IEEE

Since we employ the results of [11] based on Lyapunov-Krasowskii functionals, our conditions represent an alternative to the conditions derived using Razumikhin-like theorems in [17], [10]. Other conditions could be derived using recent results on input-to-state stability of time-delay systems via Lyapunov-Krasowskii functionals ([15], and [5] where a few comments in this regard have been presented). However, this investigation is beyond the scope of the paper.

In the next section, we present a few preliminaries, such as the definition of the quantizer and the notion of solution we adopt. The main result along with the standing assumptions are examined in Section III. Conclusions are drawn in Section IV.

Due to space limitations we can not give the proof of the result. The interested reader can find it in the unabridged version of the paper available on-line ([3]).

Notation, definitions

- $\mathbb{R}_{\geq 0}$ (respectively, $\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers.
- Let r_1, r_2 be two real numbers such that $r_1 < r_2$. Let $C^1([r_1, r_2], \mathbb{R}^m)$ (respectively, $\overline{C}^1([r_1, r_2],\ \mathbb{R}^m)$) denote the set of continuously differentiable (respectively, piece-wise continuously differentiable) functions $\phi(\cdot) : [r_1, r_2] \to \mathbb{R}^m$.
- *Norms.* $|\cdot|$ stands for the Euclidean norm, $||\phi||_c = \sup_{t\in[r_1,r_2]} |\phi(t)|$ stands for the norm of a function $\phi \in C^1([r_1, r_2], \mathbb{R}^m)$.
- $\mathrm{sgn}(r)$, $r \in \mathbb{R}$, denotes the sign function, i.e. the function such that $\mathrm{sgn}(r) = 1$ if $r > 0$, $\mathrm{sgn}(r) = -1$ if $r < 0$, and $\mathrm{sgn}(r) = 0$ if $r = 0$.
- To simplify the notation we will frequently use the notation of the Lie derivative. More precisely, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field and $h : \mathbb{R}^n \to \mathbb{R}$ is a scalar function, we may use the notation $L_f h(x)$ for $\frac{\partial h}{\partial x}(x) f(x)$.
- A continuous function $k : [0, \infty) \to [0, \infty)$ is of *class* $\mathcal{K}$ provided it is zero at zero and strictly increasing. A *class* $\mathcal{K}_\infty$ function is a *class* $\mathcal{K}$ function which in addition is unbounded.
- We shall often omit arguments of functions to simplify notation.
- For a real-valued function $z(t)$, we denote by $z(t^+)$ the right limit $\lim_{m>t, m\to t} z(m)$.

II. Problem formulation

We are interested in investigating the stability property of systems when the feedback control law undergoes quantization and delays. This problem arises in (idealized) scenarios in which a finite bandwidth channel lies in the feedback loop and introduces a delay. In the sub-sections below, we recall what is meant by quantization and what is a quantizer, we introduce the quantized time-delay system and the notion of solution we adopt, and finally the formulation of the problem.

A. Quantizers

To the purpose of describing our system in more formal terms, we introduce the following multi-valued map, which

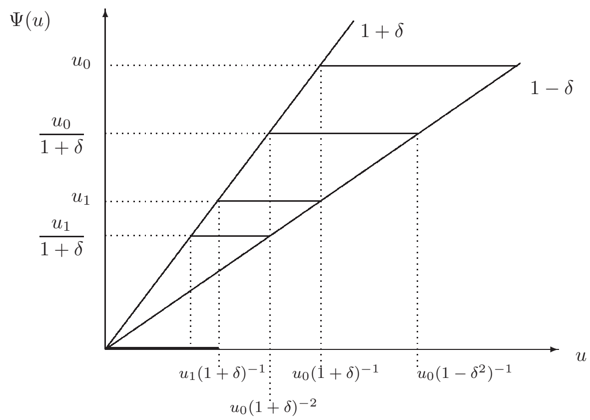

Fig. 1. The multi-valued map $\Psi(u)$ for $u > 0$, and with $j = 1$.

will be referred to henceforth as the *quantizer*. Let $u_0 > 0$ and $0 < \rho < 1$ be real numbers, let $u_i = \rho^i u_0$ and $U = \{0, \pm u_i, \pm u_i(1+\delta)^{-1},\ i = 0, 1, \ldots, j\}$, with $j \geq 1$ an integer. Let $\delta = (1-\rho)(1+\rho)^{-1}$ and

$$\Psi(u) = \begin{cases} u_i \mathrm{sgn}(u) & \frac{u_i}{1+\delta} < |u| \leq \frac{u_i}{1-\delta}\ , 0 \leq i \leq j \\ \frac{u_i}{1+\delta}\mathrm{sgn}(u) & \frac{u_i}{(1+\delta)^2} < |u| \leq \frac{u_i}{(1+\delta)(1-\delta)}\ , \\ & \qquad 0 \leq i \leq j \\ 0 & 0 \leq |u| \leq \frac{1}{1+\delta} u_j\ . \end{cases} \tag{1}$$

A picture of the map is given in Fig. 1. Observe for later use that

$$\rho = \frac{1-\delta}{1+\delta} \tag{2}$$

and

$$u_i = \left(\frac{1-\delta}{1+\delta}\right)^i u_0\ ,\ \forall i \in \{0, 1, \ldots, j\}\ . \tag{3}$$

A few remarks are in order:

- The *range* of the quantizer, i.e. its interval of definition, is $[-\frac{u_0}{1-\delta}, \frac{u_0}{1-\delta}]$. We do not define $\Psi(u)$ for $|u| > \frac{u_0}{1-\delta}$, since we will design the parameter u_0 in such a way that the control $|u(t-\tau)|$, which is the actual argument of the map Ψ, never exceeds this upper bound.
- The logarithmic quantizer with a *finite* number of quantization levels, which is a truncated version of the quantizer with an infinite number of quantization levels, was introduced in [4], Section V, and it is as follows:

$$\Psi(u) = \begin{cases} u_i \mathrm{sgn}(u) & \frac{u_i}{1+\delta} < |u| \leq \frac{u_i}{1-\delta}\ , 0 \leq i \leq j \\ 0 & 0 \leq |u| \leq \frac{1}{1+\delta} u_j\ . \end{cases} \tag{4}$$

Compared with (4), the quantizer (1) considered in this paper has additional quantization levels. To have a pictorial representation of the quantizer (4), one can refer to Fig. 1 and remove the quantization levels labeled as $\frac{u_0}{1+\delta}$ and $\frac{u_1}{1+\delta}$. The new quantization levels in (1) are added to avoid chattering (see the Remark in Subsection II-B below).

- The parameter ρ can be viewed as a measure of the quantization *density*, since the smaller is ρ, the coarser is the quantizer ([4]). In fact, by (2), as ρ approaches 0, δ approaches 1, that is the width of the sector bound in Fig. 1 gets larger and, given an interval of fixed length on the u-axis in Fig. 1, $\Psi(u)$ will have fewer quantization levels as u ranges over that interval.
- In the quantizer (1), the parameters δ, u_0, j appear. Throughout the paper, we shall assume that δ can take any value in the interval $(0,1)$ (i.e. the quantization density can be equal to any value). On the other hand the positive real number u_0 (which defines the range of the quantizer) and the integer j (which gives the number of quantization levels) are to be designed. Although it would be more correct to denote explicitly the dependence of Ψ on u_0, j, i.e. to have $\Psi_{j,u_0}(u)$, this is not pursued in the paper to avoid cumbersome notation.

B. Quantized time-delay systems

We are interested in investigating the stability of the quantized time-delay system

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(u(t-\tau)) \,, \tag{5}$$

with $x(t) \in \mathbb{R}^n$, $n \geq 1$, $f(x), g(x)$ locally Lipschitz functions, and τ a positive real number, when $u(t) = z(x(t))$, with $z(\cdot)$ a continuously differentiable real-valued function to be designed. Since $\Psi(u(t-\tau))$ is a multi-valued function, we must specify the rule by which $\Psi(u(t-\tau))$ takes value in U depending on its argument $u(t-\tau)$.

Consider the initial condition $\varphi \in C^1([-2\tau, 0], \mathbb{R}^n)$ and let $T < \tau$ be a suitable positive number. For $t \in [0,T)$ we focus our attention on $\Psi(\bar{z}(t))$, where to ease the notation we have set $\bar{z}(t) := z(\varphi(t-\tau))$. At time $t = 0$, depending on $|\bar{z}(0)|$, the value taken by the quantizer is specified as follows:

$$\Psi(\bar{z}(0)) = \begin{cases} u_i \operatorname{sgn}(\bar{z}(0)), & \frac{1}{1+\delta}u_i < |\bar{z}(0)| \leq \frac{1}{1-\delta}u_i, \; 0 \leq i \leq j \\ 0, & 0 \leq |\bar{z}(0)| \leq \frac{1}{1+\delta}u_j \,. \end{cases} \tag{6}$$

For all $t \in [0,T)$, we describe the law by which $\Psi(\bar{z}(t))$ evolves as the argument $\bar{z}(t)$ varies. Before that, in order to have a concise description, we rename the quantization levels as follows:

$$\tilde{u}_k := \begin{cases} u_{k/2} & k \text{ even} \\ \frac{u_{(k-1)/2}}{1+\delta} & k \text{ odd}, \qquad k = 0, 1, \ldots, 2j+1 \,, \end{cases}$$

and moreover we set $\tilde{u}_{2j+2} := 0$. The evolution of $\Psi(\bar{z}(t))$ obeys the law below (a pictorial representation of the law is given by the directed graph in Figure 2), where the symbol $\wedge$ denotes the logical conjunction 'and':

$$\begin{aligned} &|\Psi(\bar{z}(t))| = \tilde{u}_k \wedge |\bar{z}(t)| = \frac{\tilde{u}_k}{1+\delta} \Rightarrow |\Psi(\bar{z}(t^+))| = \tilde{u}_{k+1}, \\ &\qquad\qquad\qquad\qquad\qquad \text{for } k = 0, 1, \ldots, 2j+1 \\ &|\Psi(\bar{z}(t))| = \tilde{u}_k \wedge |\bar{z}(t)| = \frac{\tilde{u}_k}{1-\delta} \Rightarrow |\Psi(\bar{z}(t^+))| = \tilde{u}_{k-1}, \\ &\qquad\qquad\qquad\qquad\qquad \text{for } k = 1, 2, \ldots, 2j+1 \\ &|\Psi(\bar{z}(t))| = \tilde{u}_k \wedge |\bar{z}(t)| = \tilde{u}_{k-1} \Rightarrow |\Psi(\bar{z}(t^+))| = \tilde{u}_{k-1}, \\ &\qquad\qquad\qquad\qquad\qquad \text{for } k = 2j+2. \end{aligned} \tag{7}$$

If no one of the conditions on the right-hand side of the implications above is satisfied, then $\Psi(\bar{z}(t^+)) = \Psi(\bar{z}(t))$. We now specify the solution we adopt for the system

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(\bar{z}(t)) \tag{8}$$

with $t \in [0,T)$. Set $t_0 = 0$, let $\Psi(\bar{z}(t_0))$ be as in (6), compute $\Psi(\bar{z}(t_0^+))$ according to (7) above, and consider the solution $x(t)$ of

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(\bar{z}(t_0^+)) \tag{9}$$

starting from the initial condition $x_0 = \varphi(0)$, on the interval $[t_0, t_1]$, where t_1 is a time at which $\bar{z}(t)$ satisfies one of the conditions which forces $\Psi(\bar{z}(t))$ to take a new value, provided that the solution of (9) can be extended up to t_1. By definition, $\Psi(\bar{z}(t)) = \Psi(\bar{z}(t_0^+))$ for all $t \in [t_0, t_1]$, and on $[t_0, t_1]$, $x(t)$ is equivalently the solution of (8). Then, set $x_1 = x(t_1)$, compute $\Psi(\bar{z}(t_1^+))$, and consider the solution of

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(\bar{z}(t_1^+)) \tag{10}$$

starting from x_1, and defined on $[t_1, t_2]$, where t_2 is a time at which a new transition occurs. Iterating this argument, one finds a sequence $t_0, t_1, \ldots, t_k, t_{k+1}$ (for some integer $k \geq 0$, and where we have conventionally set $t_{k+1} = T$) of *switching* times, and the solution $x(t)$ of (8) on $[0,T)$ is a $\overline{C}^1$ function of time such that, for each $i = 0, 1, \ldots, k$, for all $t \in [t_i, t_{i+1})$, it satisfies

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(\bar{z}(t_i^+)) \,.$$

Remark. We now explain why chattering is avoided thanks to the introduction of additional levels in the quantizer (see also [6]). As a matter of fact, by the definition of (1), each time $\Psi(\bar{z}(t))$ makes a transition from one value to another, some (dwell) time will elapse before a new transition can occur[1]. This can be illustrated with the help of Fig. 1, where u is replaced by $\bar{z}(t)$. Suppose that, at time t, $\Psi(\bar{z}(t)) = u_0$ and $\bar{z}(t)$ hits the point $\frac{u_0}{1+\delta}$. Then $\Psi(\bar{z}(t))$ takes the new value $\frac{u_0}{1+\delta}$ (see Fig. 1). After the switching, the function $\bar{z}(t)$ can increase and eventually hits the point $u_0(1-\delta^2)^{-1}$, or decrease and eventually hits the point $u_0(1+\delta)^{-2}$. In either case, before a new transition takes place, some time will elapse, because the function $\bar{z}(t)$ must cover an interval of finite length with finite speed. In fact, for a given C^1 initial

[1]For some classes of nonlinear systems, it is possible to estimate a lower bound on such a dwell time ([2]). This is particularly important in the case in which the quantized controller is implemented over a network, since it gives indications on the data-rate needed to transmit the quantized information.

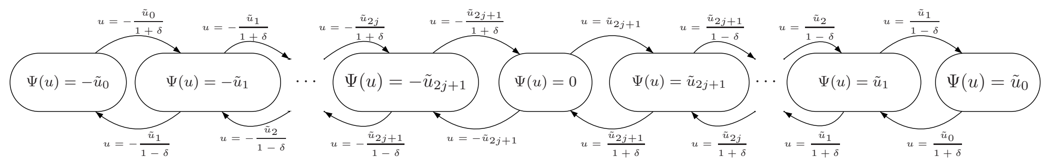

Fig. 2. The directed graph illustrates the law (7) which describes the evolution of $\Psi(u(t))$ as $u(t) = \bar{z}(t)$ varies. Each edge connects two nodes, and is labeled with the condition which triggers the transition from the starting node to the destination node.

condition φ, the time derivative of $\bar{z}(t) = z(\varphi(t-\tau))$ is continuous and bounded on $[0, T)$, and in particular:

$$|\frac{d\bar{z}(t)}{dt}| \le \max_{|x|\le R} |\frac{\partial z(x)}{\partial x}| \cdot \max_{t\in[-2\tau,-\tau]} |\frac{d\varphi(t)}{dt}| \; .$$

If, on the other hand, we were adopting the quantizer (4), $\Psi(\bar{z}(t))$ would have taken the value u_1 rather than $\frac{u_0}{1+\delta}$. Immediately after the switching, it could happen that $\bar{z}(t)$ cannot decrease, thus forcing a transition to the previous value, which would in turn trigger a new transition to u_1, and this would continue to happen again and again. It is precisely to avoid such fast transitions that new quantization levels were added. This addition can be seen as a way to add *hysteresis* to the quantized system, and we will refer to (1) as a quantizer with hysteresis.

For the analysis to follow, the following observation is important. For each $t \in [0, T)$, such that $t \in [t_i, t_{i+1})$, $i = 0, 1, \dots, k$, if $|\bar{z}(t)| < u_0(1-\delta)^{-1}$, then the solution $x(t)$ of (8) satisfies the differential inclusion

$$\dot{x}(t) \in f(x(t)) + g(x(t))K(\Psi(\bar{z}(t))) \; , \tag{11}$$

where $K(\Psi(u))$, with $u = \bar{z}(t)$, is such that

$$K(\Psi(u)) \subseteq \begin{cases} \{v \in \mathbb{R} : v = (1+\lambda\delta)u \,,\, \lambda \in [-1, 1]\} \,, \\ \qquad (1+\delta)^{-1}u_j < |u| \le (1-\delta)^{-1}u_0 \\ \{v \in \mathbb{R} : v = \lambda(1+\delta)u \,,\, \lambda \in [0, 1]\} \,, \\ \qquad |u| \le (1+\delta)^{-1}u_j \; . \end{cases} \tag{12}$$

This is easily verified bearing in mind that, by the definition (1) of the map $\Psi(u)$, $\Psi(u) \in K(\Psi(u))$ for all $|u| < u_0(1-\delta)^{-1}$.

C. Problem formulation

Because the quantizer outputs a zero value in the vicinity of the origin, asymptotic stability of the origin of (5) is not possible to achieve (except in exceptional cases without interest). We are rather interested in the following property:

Definition. The system

$$\dot{x}(t) = f(x(t)) + g(x(t))v(t-\tau) \; , \tag{13}$$

with $\tau \ge 0$ is semi-globally practically stabilizable by quantized feedback if for any $\varepsilon < R < 0$ there exist a law $z(x)$, a real number $u_0 > 0$ and an integer $j \ge 1$ such that the solution of

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi(z(x(t-\tau))) \; , \tag{14}$$

starting from $\mathcal{R} = \{\varphi \in C^1([-2\tau, 0], \mathbb{R}^n) : ||\varphi||_c \le R\}$ enters B_ε, the closed ball of radius ε, at some finite time $t_s \ge 0$, and remains in that set for all $t \ge t_s$.

In the remaining sections, we propose a solution to the problem formulated above.

III. Standing assumptions and main result

A. Basic assumptions

The result to be derived below for the system (13) holds under the following standing assumptions.

(A1) There exist a continuously differentiable positive definite and proper Lyapunov function $V(x)$, two class $\mathcal{K}_\infty$ functions κ_1, κ_2, a positive definite continuous function $W(x)$ and a continuously differentiable real-valued function $z(x)$, which is zero at the origin, with $W(x)$ and $z(x)$ both depending on δ, such that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &\kappa_1(|x|) \le V(x) \le \kappa_2(|x|) \; , \\ &\frac{\partial V}{\partial x}[f(x) + g(x)(1+p)z(x)] \le -W(x) \; , \quad p \in [-\delta, \delta] \; . \end{aligned} \tag{15}$$

Remark. The uncertainty in the input channel is modeled through the parameter p, whose range depends on the quantization density through δ. Such uncertainty takes into account the effect due to quantization, as it should be evident from (12). Assumption (A1) amounts to require the system $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, with *no* delay, to be stabilizable in the presence of quantization. The design of a stabilizing quantized feedback is carried out e.g. in [1] (see also Subsection III-B below).

The next two assumptions require the system to be robust with respect to delays. In particular they are needed to guarantee that no finite-escape time phenomenon will occur, and that the solution stays bounded for all the times. These conditions also appear in [11] (where no quantization was present), although in a slightly different form. The difference is due to the fact that the quantization effect adds up to the delay effect, and in the conditions below also the quantization parameter δ plays a role. For more comments on these two assumptions the Reader is referred to [3].

(A2) Let Ω be a positive real number which satisfies $\Omega \ge 16\tau$. For all $x \in \mathbb{R}^n$, for all $\xi \in \overline{C}^1([0, 2\tau], \mathbb{R}^n)$, for all $\lambda_1 \in [-1, 1]$ and for all $\lambda_2 \in \overline{C}^0([0, \tau], \mathbb{R})$ such that $\lambda_2(m) \in [1-\delta, 1+\delta]$ for all $m \in [0, \tau]$, the inequality

$$-\frac{1}{4}W(x) - T(x, \xi, \lambda_1, \lambda_2) - \frac{1}{\Omega}\int_0^{2\tau} W(\xi(\ell))d\ell \le 0 \; , \tag{16}$$

with

$$T(x,\xi,\lambda_1,\lambda_2) = L_gV(x)(1+ \\ +\lambda_1\delta)\int_{\tau}^{2\tau} H(\xi(\ell),\xi(\ell-\tau),\lambda_2(\ell-\tau))d\ell \ ,$$

$$H(a,b,c) = L_f z(a) + L_g z(a) c z(b) \ ,$$

holds.

(A3) There exists a nondecreasing function $\kappa_3(\cdot)$ of class C^1 such that for all $x \in \mathbb{R}^n$, for all $L \geq 0$ and for all $\lambda \in [-1,1]$, the inequality

$$-\frac{1}{2}W(x) + \sup_{|a|\leq L}\{L_gV(x)(1+\lambda)[z(a)-z(x)]\} \leq \\ \leq \kappa_3(L)[V(x)+1] \tag{17}$$

holds.

B. Comments on the Assumption (A1)

A number of ways to have Assumption (A1) fulfilled are discussed below.

Lyapunov Redesign. Suppose that, for the system (13), are known a function V of class C^2, and a function $\zeta(x)$ of class C^1 such that, instead of (15), only the weaker condition

$$L_fV(x) + L_gV(x)\zeta(x) = -\tilde{W}(x) \ , \tag{18}$$

with $\tilde{W}(x)$ a continuous positive definite function, is satisfied. Introduce the control law

$$z(x) = \zeta(x) - \alpha(x)L_gV(x) \ , \tag{19}$$

with $\alpha(x)$ a positive function to be chosen later. Then we have

$$\begin{aligned} &\frac{\partial V}{\partial x}[f(x)+g(x)(1+p)z(x)] \\ &= \frac{\partial V}{\partial x}[f(x)+g(x)\zeta(x)] - \alpha(x)\left|L_gV(x)\right|^2 + \\ &\qquad +pL_gV(x)\left[\zeta(x)-\alpha(x)L_gV(x)\right] \\ &\leq -\tilde{W}(x) - \alpha(x)(1+p)\left|L_gV(x)\right|^2 + pL_gV(x)\zeta(x) \\ &\leq -\tilde{W}(x) - \alpha(x)(1-\delta)\left|L_gV(x)\right|^2 + \delta\left|L_gV(x)\right|\left|\zeta(x)\right| \ . \end{aligned}$$

A simple completion-of-the-squares argument shows that

$$\frac{\partial V}{\partial x}[f(x)+g(x)(1+p)z(x)] \leq -\frac{3}{4}\tilde{W}(x) \ ,$$

provided that

$$\alpha(x) \geq \frac{\delta^2}{1-\delta}\frac{|\zeta(x)|^2}{\tilde{W}(x)} \ . \tag{20}$$

Hence, the control law (19), with $\alpha(x)$ defined above and such that $\lim_{x\to 0}\alpha(x)L_gV(x) = 0$, guarantees the fulfillment of Assumption (A1) with $W(x) = 3\tilde{W}(x)/4$.

Sontag's universal stabilizer [16]. Consider the system

$$\dot{x} = f(x) + g(x)[1+p]u \ , \tag{21}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $p \in [-\delta,\delta]$, $\delta \in [0,1)$. Let us assume that a control Lyapunov function $V(x)$ is known for the system (21) with $p = 0$, and set

$$\dot{V}(x) = a(x) + [1+p]b(x)u \ , \tag{22}$$

with $a(x) = L_fV(x)$, $b(x) = L_gV(x)$. Since V is a control Lyapunov function for (21) with $p = 0$, $b(x) = 0$ implies $a(x) < 0$ when $x \neq 0$. Next, consider the control given by Sontag's formula:

$$\begin{aligned} &u(x) = K\tfrac{-a(x)-\sqrt{a(x)^2+b(x)^4}}{b(x)} \text{ when } b(x)\neq 0 \ , \\ &u(x) = 0 \text{ when } b(x) = 0 \ , \end{aligned} \tag{23}$$

and where K is a positive real number to be selected later. Then, when $b(x) \neq 0$, the derivative of V along the trajectories of (21) in closed-loop with $u(x)$ defined in (23) satisfies

$$\begin{aligned} \dot{V}(x) &= a(x) + [1+p]b(x)K\frac{-a(x)-\sqrt{a(x)^2+b(x)^4}}{b(x)} \\ &= a(x) - [1+p]Ka(x) - [1+p]K\sqrt{a(x)^2+b(x)^4} \\ &= [1-(1+p)K]a(x) - [1+p]K\sqrt{a(x)^2+b(x)^4} \ . \end{aligned} \tag{24}$$

We choose $K = \frac{2}{1-\delta} > 0$. Then, when $a(x) \geq 0$, we have

$$\dot{V}(x) \leq -a(x) - [1+p]K\sqrt{a(x)^2+b(x)^4} \ , \tag{25}$$

and, when $a(x) < 0$,

$$\dot{V}(x) = a(x) - [1+p]K(a(x) + \sqrt{a(x)^2+b(x)^4}) < 0 \ . \tag{26}$$

When $b(x) = 0$, then

$$\dot{V}(x) \quad = \quad a(x) < 0 \text{ if } x \neq 0 \ . \tag{27}$$

Under the *small control property* ([16]) one can prove that the control law introduced above is smooth everywhere except at the origin where it may be only continuous. However, in many cases, the control law turns out to be also continuously differentiable at the origin, and then a continuously differentiable function $z(x)$ which guarantees the inequality (15) is obtained.

Lyapunov stable systems. Consider again system (21), and assume that a Lyapunov function $V(x)$ such that $a(x) \leq 0$, and $b(x) \neq 0$ when $x \neq 0$ and $a(x) = 0$, is known. Then, selecting $u = -\xi(x)b(x)$, where ξ is any C^1 positive function, we obtain, for all $x \in \mathbb{R}^n$

$$\dot{V}(x) \leq a(x) - [1-\delta]\xi(x)b(x)^2 \tag{28}$$

and the function $a(x) - [1-\delta]\xi(x)b(x)^2$ is negative definite. Inequality (15) then holds with $z(x) = -\xi(x)b(x)$ and $W(x) = -a(x) + [1-\delta]\xi(x)b(x)^2$.

Dissipation inequality [6], [1]. Consider the system (13). Suppose that a Lyapunov function $V(x)$ is known such that for all $x \in \mathbb{R}^n$

$$L_fV(x) - \frac{1}{4}(1-\delta^2)\left(L_gV(x)\right)^2 \leq -\tilde{W}(x) \ .$$

Then, for any $p \in [-\delta,\delta]$ it is also true that

$$L_fV(x) - \frac{1}{4}(1-p^2)\left(L_gV(x)\right)^2 \leq -\tilde{W}(x) \ .$$

Define now $z(x) = -\dfrac{1}{2}L_gV(x)$ and observe that the inequality above rewrites as

$$L_f V(x)+\frac{1}{4}p^2\left(L_g V(x)\right)^2+z(x)L_g V(x)+z(x)^2 \leq -\tilde{W}(x) \ ,$$

or, equivalently,

$$\frac{\partial V}{\partial x}(f(x)+g(x)z(x))+\frac{1}{4}p^2\left(L_g V(x)\right)^2+z(x)^2 \leq -\tilde{W}(x) \ . \tag{29}$$

We remark incidentally ([1]) that the latter inequality implies the existence of a control $u = z(x)$ which renders the system

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + g(x)w \ , \\ z & = & u \ , \end{array} \right.$$

strictly dissipative with respect to the supply rate $q(w, z) = -z^2 + p^{-2}w^2$.

Observe now that

$$pz(x)L_g V(x) \leq \frac{1}{4}p^2\left(L_g V(x)\right)^2 + z(x)^2$$

and therefore (29) implies that

$$\frac{\partial V}{\partial x}(f(x) + g(x)z(x)) + pz(x)L_g V(x) \leq -\tilde{W}(x) \ ,$$

that is (15) with $W(x) = \tilde{W}(x)$.

C. Main result

We are ready to state the main result of our work. As already made clear in the problem formulation (see Definition in Subsection II-C), the two main design parameters are the range u_0 and the number j of levels of the quantizer. Intuitively, to design u_0 we need to quantify the "overshoot" of the state variable and we expect this to depend on the size of the initial condition. Regarding the number of quantization levels j, it is not hard to figure out that in general the closer one wants to confine the state to the origin (i.e. the smaller ε is in the Definition in Section II-C), the larger the number of quantization levels must be. On the other hand, having fixed the width of the quantizer, the number of the quantization levels will increase with the range u_0 and in turn with R. Such a dependence is made clear in the statement below.

Proposition 1: Let us assume that the system (13) satisfies Assumptions (A1) to (A3). Then the origin of (13) is semi-globally practically stabilizable by quantized feedback. Namely, there exist a positive, continuous and non-decreasing function $u_0(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$, and a positive continuous function $j(\cdot,\cdot) : \mathbb{R}^2_{\geq 0} \rightarrow \mathbb{R}_{>0}$ such that, for any $R > \varepsilon > 0$, if $u_0 \geq u_0(R)$, $j \geq j(\varepsilon, R)$ and z is the feedback provided by Assumption (A1) satisfying (15), then the solution of (14) starting from $\mathcal{R} = \{\varphi \in C^1([-2\tau, 0], \mathbb{R}^n) : ||\varphi||_c \leq R\}$ enters B_ε, the closed ball of radius ε, at some finite time $t_s \geq 0$, and remains in that set for all $t \geq t_s$.

The proof is omitted and can be found in [3]. A few comments are in order. The proof is based on a Lyapunov-Krasowskii functional given by the sum of the Lyapunov function $V(x)$ in Assumption (A1) and a term which at time t depends on the state $x(\cdot)$ restricted to the interval $[t-2\tau, t]$. Hence, in order to use such a Lyapunov-Krasowskii functional, we need to first prove that all solutions of the closed-loop system we consider exist for all $t \in [-2\tau, 2\tau]$. To this purpose, we only make use of the Lyapunov function $V(x)$. Then we prove that the solutions can be extended beyond 2τ, showing that the Lyapunov-Krasowskii functional is bounded for all the time and finally that the solutions converge in finite time to a ball around the origin of radius ε. The proof is constructive and provides the explicit expression for the quantizer parameters u_0 and j.

IV. Conclusion

We have presented a Lyapunov-Krasowskii functional approach to solve the problem of determining quantized feedbacks with delay which semi-globally practically stabilize the origin of nonlinear systems. For a fairly general family of systems, and given any value of the quantization density, we have characterized the maximal allowable constant delay which the closed-loop system can tolerate. A problem which in our opinion would be interesting to investigate is how, for systems with a well-defined relative degree, our result can be propagated via the backstepping technique.

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